This article appeared in a journal published by Elsevier. The attached copy is furnished to the author for internal non-commercial research and education use, including for instruction at the authors institution and sharing with colleagues.

Other uses, including reproduction and distribution, or selling or licensing copies, or posting to personal, institutional or third party websites are prohibited.

In most cases authors are permitted to post their version of the article (e.g. in Word or Tex form) to their personal website or institutional repository. Authors requiring further information regarding Elsevier’s archiving and manuscript policies are encouraged to visit:

http://www.elsevier.com/copyright
Parameterization of particle transport at submesoscales in the Gulf Stream region using Lagrangian subgridscale models

Angelique C. Haza\textsuperscript{a, *}, Tamay M. Ö zgökmen\textsuperscript{a}, Annalisa Griffa\textsuperscript{a,b}, Zulema D. Garraffo\textsuperscript{c}, Leonid Piterbarg\textsuperscript{d}

\textsuperscript{a} MPO/RSMAS, University of Miami, 4600 Rickenbacker csw, Miami, Florida 33149-1098, USA
\textsuperscript{b} ISMAR/CNR, U.O.S di Pizzoallo di Lerici (SP), Forte Santa Teresa, 10932 Lerici (SP), Italy
\textsuperscript{c} MPO/RSMAS, University of Miami, 4600 Rickenbacker csw, Miami, Florida 33149-1098, USA
\textsuperscript{d} University of Southern California, 3620 South Vermont Ave., KAP 108, Los Angeles, California 90089-2532, USA

\section*{1. Introduction}

It is now common for many applied problems, such as oil spills, dispersion of pollutants, and biogeochemical transport to make use of the Lagrangian framework:

\begin{equation}
\frac{dr}{dt} = \mathbf{v}(t) = \mathbf{u}(\mathbf{r}, t),
\end{equation}

where \( \mathbf{r} \) is the space vector, \( \mathbf{v}(t) \) is the temporal evolution of the Lagrangian velocity vector of a particle along its trajectory and \( \mathbf{u}(\mathbf{r}, t) \) is the corresponding Eulerian velocity field. Since it has been long understood that even simple time dependence of turbulent coherent structures can create chaotic trajectories of individual particles (Aref, 1984), a large number of particles is needed to carry out a statistical analysis of transport.

Both single and dual particle statistics are considered here. Single particle metrics include the absolute dispersion \( \rho \) defined as:

\begin{equation}
\rho(t) = \left( \langle |\mathbf{r}(t) - \mathbf{r}(0)|^2 \rangle \right)^{1/2},
\end{equation}

and the second moment of particle displacement defined as:

\begin{equation}
\sigma_{Cu}(t) = \left( \langle |\mathbf{r}(t) - \langle \mathbf{r}(t) \rangle |^2 \rangle \right)^{1/2},
\end{equation}

or dispersion around the center of mass.

Relative (two-particle) dispersion, which is directly connected to turbulent and scalar fluctuations in the underlying flow field is given by

\begin{equation}
D^2(t) = \left( \langle |\mathbf{r}_1(t) - \mathbf{r}_2(t)|^2 \rangle \right).
\end{equation}

Their kinematic relation is established by considering the absolute and relative dispersion coefficients, respectively (Babiano et al., 1990):

\begin{equation}
K(t) = \frac{1}{2} \frac{d}{dt} \rho^2(t), \ Y(t, D_0) = \frac{1}{2} \frac{d}{dt} D^2(t, D_0),
\end{equation}

where \( D_0 \) is the initial separation distance of particle pairs, and leading to:
lasting coherent flow features. Second, progress was achieved in features in rapidly-rotating, strong-stratified flows lead to long-

ments in dynamical systems techniques. These techniques locate distinguished finite-time invariant flow boundaries in the Lagrang-

in time particle advection. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this paper.)

Another diagnostic closely related to relative dispersion is the finite scale Lyapunov exponent, $\lambda(\delta)$ or FSLE (Artale et al., 1997; Aurell et al., 1997):

$$Y(t, D_0) = 2K(t) - 2 \int_0^t \langle \mathbf{V}(r_1, t) \mathbf{V}(r_2, \tau) \rangle d\tau + \langle D_0 \delta \mathbf{V}(t, D_0) \rangle,$$

where $\delta \mathbf{V} = d\mathbf{D}/d\tau$ is the vector of Lagrangian relative velocity. Thus the relative diffusivity is the sum of three contributions: the first term is twice the absolute diffusivity, the second term is the cross-correlation of pair velocities, and the third term is equal to zero in homogeneous turbulence.

Provided that the task of launching an adequate number of drifters to capture most pertinent degrees of freedom in ocean flows is often restricted by prohibitive costs and environmental concerns, the computation of $D^2(t)$ or $\lambda(\delta)$ has been a challenging matter. Nevertheless, two major areas of progress have been attained over the past two decades. First, understanding of the geometry of mixing and transport in the ocean has greatly benefited from the developments in dynamical systems techniques. These techniques locate distinguished finite-time invariant flow boundaries in the Lagrangian frame, by computing so-called Lagrangian coherent structures (LCS) (Haller and Poje, 1998; Haller, 2001; Wiggins, 2005; Shadden et al., 2005), and have been shown to be quite useful in the context of mesoscale ocean flows, for which the high aspect ratio of the features in rapidly-rotating, strong-stratified flows lead to long-lasting coherent flow features. Second, progress was achieved in the development of realistic ocean general circulation models (Chassignet et al., 2006; Capet et al., 2008; Martin et al., 2009) that capture explicitly, and/or through satellite altimeter data assimilation, the turbulent features of the ocean resulting from the baroclinic and barotropic instabilities of the main circulation, namely mesoscale eddies and jets. As such, realistic ocean and coastal models are often used to generate large ensembles of synthetic trajectories.

The LCS approach is fully deterministic, namely it relies on the precise knowledge of the Eulerian velocity field $u(x, t)$ at all scales of motion to integrate Eq. (1). While computation of LCS-based transport in the mesoscale range from velocity fields derived from quasi-geostrophic analysis of satellite surface data (Abraham and Bowen, 2002; Waugh et al., 2006; Beron-Vera et al., 2008) and/or directly from OGCMs (d’Ovidio et al., 2004; Olascoaga et al., 2006; Haza et al., 2007a; Mancho et al., 2008) has a solid basis, Griffa et al. (2004) highlighted the fact that there continues to exist (and will continue to exist) a significant gap between the spatial and temporal scales resolved by even the highest resolution ocean model and those scales affecting the motion of Lagrangian particles in the actual ocean. It has been recently recognized that various submesoscale processes in the scale range of 10 km to 100 m can play a critical role in the energy cascade of the ocean (Müller et al., 2005; McWilliams, 2008) and biogeochemical transport (Klein and Lapeyre, 2009). Langmuir circulations arising from the interaction between the Stokes drift due to surface waves and the vertical shear of the wind-induced currents (Craik and Leibovich, 1976) can be challenging to capture even in the most complete numerical models (Skillingstad and Denbo, 1995). In addition, uncertainties and lack of temporal and space resolution

![Fig. 1. FSLE branches from 1/12° (upper panel) and 1/48° (lower panel) HYCOM simulations in the Gulf Stream region. Note the rich submesoscale field in the higher resolution case. The color panels indicate FSLE in 1/day. Blue colors show inflowing/stable LCS from forward in time, and red colors out-flowing/unstable LCS from backward in time particle advection. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this paper.)](image-url)
exist in air-sea interaction data that are used as forcing. A question thereby arises on the possible impact of unresolved scales of motion on Lagrangian transport.

On the basis of a hierarchy of numerical models, Poje et al. (2010) showed that hyperbolicity in oceanic horizontal flow fields increases as the mesh size reduces. The hyperbolicity at the smallest resolved scales can be quantified using an Eulerian metric, namely the positive (strain dominated) partition of the Okubo-Weiss criterion, which is found to be directly proportional to the maximum rate of exponential separation experienced in ensembles of particles. Scale-dependent FSLE \( \lambda (s) \) plots showed that this increase occurs only in the submesoscale range while the mesoscale transport remains unchanged, that is, for the scales larger than the first baroclinic Rossby deformation radius \( R_d \). The implication is that as the model resolution increases, submesoscale filaments and eddies act to enhance particle pair separation at their scale range.

One of the proxies for LCS is FSLE maps computed from forward and backward in time advection of synthetic particles:

\[
\lambda (r) = \frac{\ln (d_r/d_i)}{\tau_r (d, d_f)},
\]

where \( d_i \) is the initial separation distance fixed at about half the grid-spacing, and \( \tau_r \) is the time it takes the local particle pair to reach the distance \( d_r \). An example is given in Fig. 1 (for \( d_f = 20d_i \), showing the FSLE maps of two HYCOM numerical simulations of the Gulf Stream with grid spacings of 1/12\( ^\circ \) and 1/48\( ^\circ \), respectively. The lower resolution simulation displays a web of dispersion extrema corresponding to the transport barriers of the mesoscale circulation (namely, the Gulf Stream meanders and eddies). Since the grid spacing is about one fourth the \( R_d \), the model cannot really resolve the submesoscale motions. However, the 1/48\( ^\circ \) configuration can resolve features down to about 4 km, and the submesoscale flow instabilities are evidenced by an explosion of relatively short-lived, complex web of FSLE extrema. Direct comparison of scale-dependent FSLE indicates that the enhanced strain by such submesoscale features contributes to an increase in \( \lambda \) at the smallest separation scale \( \delta = 2 \text{ km} \), denoted \( \lambda_{\text{max}} \) by a factor of two (Fig. 3).

There have been very few observations tailored to sample the submesoscale dispersion. Lumpkin and Elipot (2010) found in the Gulf Stream region three distinct regimes below the radius of deformation: a Richardson regime down to 10 km, a ballistic regime down to 2 km, and indication of an exponential regime between 1 and 2 km with \( \lambda_{\text{max}} \) more than ten times higher than in HYCOM 1/48\( ^\circ \) (Fig. 3). Values as high as \( \lambda_{\text{max}} \approx 5 \text{ day}^{-1} \) were obtained also from observations in the coastal region of LaSpezia near the Italian coast from a VHF radar (Haza et al., 2010). The emergence of a ballistic regime in the submesoscale range, on the other hand, has not been observed before, even though evidences of multiple regimes occurring at the submesoscales have been reported by Berti et al. (2011) in the Southwestern Atlantic. Overall, observations results, even though still sparse and not always in agreement in the identification of specific regimes, appear to indicate that the submesoscale dispersion tends to be underestimated in OGCMs. As such, parameterizations are needed to enhance unrealistically low strain levels at the submesoscale range in order to improve transport estimations based on numerical models.

Since the OGCMs are computationally expensive and parameterizations of submesoscale effects in them still constitute an open issue (Fox-Kemper et al., 2008), we put forth an approach in which

---

**Fig. 2.** Three month mean velocity vectors and standard deviation of HYCOM 1/12\( ^\circ \) (left panel) and HYCOM 1/48\( ^\circ \) (right panel) at 15 m-depth. The velocity vectors are sampled to every 1\( ^\circ \) for both 1/12\( ^\circ \) and 1/48\( ^\circ \) configurations.
the parameterizations act directly on the Lagrangian transport Eq. (1) rather than on the OGCMs fields, allowing for greater computational efficiency. In light of the increasing complexity of the FSLE barriers in high resolution OGCMs, such Lagrangian subgridscale (LSGS) parameterizations should be of statistical nature, allowing to incorporate the net effect of turbulent motions on relative dispersion in the submesoscale range.

Here we consider and test three LSGS models. The first two belong to the class of Markovian stochastic particle models described by Griffa (1996). The first and simplest (random walk) model is Markovian in the distance, and assumes that the time scales of turbulent motions are infinitesimal. The second and more realistic model (random flight) is joint Markovian in both the distance and the velocity, and introduces a finite decorrelation time scale for the turbulent velocity. Both stochastic models have the capability to change the absolute dispersion of a particle ensemble, and should therefore affect their relative dispersion, since the two metrics are related (Babiano et al., 1990). The third model, which is derived on the basis of Markovian principles, but is not stochastic in its final form, was introduced by Haza et al. (2007b), and is further investigated here.

In this study, we present results from a unification of deterministic and statistical methods for Lagrangian transport. Dynamical system approach with un-modified velocities will be preserved for scales \( \delta \gg R_d \) while three LSGS models are tested for the submesoscale range, \( \delta < R_d \). The relative dispersion, a metric quantifying the net effect of all turbulent motions on two particle separation, or the FSLE, provide a means for precise evaluations of the performance of these parameterizations. The multi-resolution 1/12° and 1/48° surface velocity fields from HYCOM in the Gulf Stream region constitute our testbed, supplemented by recent observations by Lumpkin and Elipot (2010).

The paper is organized as follows: We discuss the computational setting and dispersion regimes of HYCOM in Section 2, and introduce LSGS models in Section 3. Relative dispersion in the presence of LSGS models are presented in Section 4. A discussion on the performance characteristics of LSGS models is provided in Section 5. Finally, conclusions are summarized in Section 6.

2. Gulf Stream region and dispersive regimes from HYCOM

2.1. Computational setting

This work uses velocities from 15 m depth in a Gulf Stream simulation. The simulation was conducted using the Hybrid Coordinate Ocean Model (HYCOM, Bleck, 2002; Chassignet et al., 2006).

Fig. 3. Scale-dependent FSLE \( \lambda(\delta) \) from HYCOM solutions with horizontal mesh spacings of 1/12° (black) and 1/48° (brown). The observational result from Lumpkin and Elipot (2010) is reproduced and shown by the dashed line. Ballistic (\( \lambda \sim \delta^{-2/3} \)) and Richardson (\( \lambda \sim \delta^{-1} \)) regimes are indicated. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this paper.)

Fig. 4. (Upper panel) Relative dispersion \( D^2(t) \). Richardson regime, \( t^3 \), ballistic regime, \( t^2 \), and diffusive regime, \( t \), are indicated in the background. (Lower panel) Absolute dispersion \( \rho(t) \) from HYCOM 1/12° and 1/48° simulations.
The Gulf Stream simulation is configured on a mercator horizontal grid with 1/48° grid size, and has 30 vertical hybrid layers, of which the top 6 are in z coordinates. The lateral boundary conditions are provided by an external solution on a 1/12° grid of HYCOM, covering the Atlantic Ocean and the Mediterranean Sea, from 28°S to 80°N (Chang et al., 2009).

The nesting method employed here is available in the standard HYCOM source. For the barotropic flow, boundary conditions following the method of characteristics are applied to the normal velocities and pressure, while the tangential velocities are imposed. For the baroclinic flow, the normal velocities and the total mass fluxes are prescribed, while the tangential velocities are nudged, and other boundary conditions like interface pressure are nudged within a finite width zone.

The external and internal model solutions are based on the same depth data set, parameters and forcing. The model topography is obtained from the Digital Bathymetric Data Base (DBDB2), from US Naval Research Laboratory (http://www7320.nrlssc.navy.mil/DBDB2_WWW/).

The thermodynamical atmospheric forcing is based on monthly values from the ECMWF 40-year reanalysis (ERA40, Uppala (2005)) for years 1978–2002. The mechanical forcing is based on the same data set, on which 6-hourly perpetual year wind stress and wind speed anomalies, derived from Navy Operational Global Atmospheric Prediction System Model, NOCAPS, for January 2003 to January 2004 are superimposed. Atmospheric forcing values are extrapolated from the ocean onto land to avoid discrepancies between atmospheric and ocean model land–sea masks (Kara et al., 2007). Several bias corrections are applied that are in use at National Research Laboratory, some of which are described for their ERA40 simulation by Metzger et al. (2010) (precipitation and the wind speed corrected through correlations with satellite observations, limiting maximum wind velocity to non-hurricane winds). Vertical turbulent mixing is based on the KPP algorithm (Large et al., 1994).

The Gulf Stream model is initialized by replacing a state obtained from the solution at the beginning of January of climatological year 8 of the Atlantic domain, on a state obtained from the Generalized Digital Environmental Model (GDEM) Version 3.0 climatology (Carnes, 2009).

Mean 15 m depth velocities for the 3 months of the simulations studied in this paper show the Gulf Stream, eddies, and recirculation features at 1/12° and 1/48° resolutions (Fig. 2 upper panel). The 1/48° simulation shows more of the Gulf Stream flow extending to the eastern part of the domain with respect to the 1/12° simulation (the circulation at the model boundaries is fixed by the 1/12° simulation). The velocity standard deviation in the 1/48° simulation (Fig. 2 lower panel) shows an energetic eddy field near the Gulf Stream separation, with maximum value of 140 cm/s, in better agreement with observations than 1/12° simulations (Garraffo et al., 2001a).

In both 1/12° and 1/48° velocity fields, a set of more than 5,000 particles are launched between Cape Hatteras and the Grand Banks. The particles are initialized as triplets that are 2.4 km apart from the central one, and the triplets are released every 4 (20) grid points in the 1/12° (1/48°) mesh. This configuration yields more than 4,000 available original pairs for the relative dispersion metrics. A fourth-order Runge–Kutta scheme is used to solve the Lagrangian equation with an integration time-step of Δt = 2 h, and a third-order polynomial scheme for spatial interpolation. Results from on-line and off-line particle advection with the same time-step of 2 h are compared, and good agreement is seen so that all subsequent tests for LSGS models are done using off-line particle advection. The total duration of the advection ranges between two to three months, depending on the time of launch. This period allows for statistical significance when computing the relative dispersion for the relevant range of scales.

2.2. HYCOM dispersion regimes

The scale-dependent FSLE is computed by using the fastest-crossing method from original triplet pairs (see Poje et al. (2010) for a thorough description). The FSLE λ(δ) in Fig. 3 displays an exponential regime for the separation scales 2 km ≤ δ ≤ 50 km, that is for the scales smaller than the radius of deformation in this region (35 km ≤ Rd ≤ 50 km). For scales larger than Rd, the curve describes a power-law regime close to a ballistic regime (λ = δ−1), albeit slightly more diffusive. The distinct transition between the two regimes at δ = Rd results from the hyperbolic regions in between the smallest resolvable eddies setting the rate of separation of particle-pairs down to the smallest scales.

While the finer scale motions of the flow do not impact the large time and space scale dispersion controlled by the energetic mesoscale features, they may have a direct impact on the submesoscale dispersion regime (so-called local dispersion, Bennett, 1984). The submesoscale eddies forming most possibly by mixed layer instabilities (Boccaletti et al., 2007; Fox-Kemper et al., 2008) or shallow baroclinic instabilities along the edges of mesoscale eddies are stretched by the mesoscale eddies and act to enhance the strain field. While the exact mechanisms of how these turbulent interactions take place await careful future investigations, their net effect on the λ(δ) plot appears as a scale-dependent (local) dispersion regime over the scale range of resolved instabilities, as well as a larger λmax. As such, the pronounced kink at δ = Rd implies the absence of submesoscale motions in HYCOM 1/12°, since the model grid spacing is too coarse to resolve them. As shown in Fig. 4 D2(t) displays an exponential regime for the first 3 days, followed by a clear Richardson regime characterized by a t3 power law. As such, there is some degree regarding the nature of power law at the mesoscale range estimated from these two metrics, requiring the use of both for evaluation purposes.

As expected from the above discussion, the relative dispersion of the Gulf Stream in HYCOM 1/48° has similar regimes for the large time and space scales, yet differs in the submesoscale range. The exponential rate of separation described by D2(t) (Fig. 4) is faster and does not exceed 2 days. It is followed by a t3 power-law up to about 2 months, then shifts to the diffusive regime (D2~t). Similarly, λ(δ) displays a much shorter exponential regime which does not extend beyond δ = 5 km, and λmax = 0.7 day−1, which is about twice the λmax from HYCOM 1/12°. The transition to the near-balistic power law occurs also at 2–3 Rd. The smoother transition from the exponential regime to the power-law regime in the intermediate submesoscale range indicates a range of finer scale motions due to smaller eddies and filaments resolved by the model.

The absolute dispersion for both HYCOMs has two distinct power-law regimes: r(δ)~t in the first few days, before transiting to the ρ~√t diffusive regime (Fig. 4). The corresponding diffusivity coefficients are K~9.3 × 10−6 m2 s−1 and K~9.8 × 10−4 m2 s−1 for 1/12° and 1/48°, respectively.

3. The LSGS models

3.1. LSGS-1 model

A zeroth-order Markov model (random walk) for two-particle motion is introduced. Given two particles at a distance of δ apart,
their random increments in the zonal direction (and analogously in the meridional direction) are computed from:

$$\begin{align*}
d_{x_1} &= L_K dw_{0_A} \\
d_{x_2} &= D dx_1 + (1 - D) L_K dw_{0_B},
\end{align*}$$

(9)

where $L_K = \sigma dt$ is a model parameter representing the standard deviation of the length scale that the particle travels under the action of turbulent processes over a time period of $dt$, $\sigma^2 = \langle u^2 \rangle$ is the turbulent velocity variance, and $u_{0_B}$ is the model (HYCOM) zonal velocity component. $D$ is a parameter representing the effect of submesoscale eddies on particle pair motion, and is defined as:

$$D = \exp \left( -\frac{\delta^2}{2L_K^2} \right),$$

(10)

where $L_K$ is the space correlation scale. For instance, assuming a turbulent velocity of $u = 0.1 \text{ m/s}$ or 10% of the typical velocity in the Gulf Stream over a duration of $dt = 2 \text{ h}$ (the period in HYCOM implementation), one gets $L_K \approx u dt \approx 0.7 \text{ km}$. Here, $dw_0$ is a random increment from a Gaussian pdf of zero mean and standard deviation of unity (the subscripts $A$ and $B$ refer to 2 distinct increments). Therefore, in the absence of a spatial correlation ($L_K = 0$), $D \rightarrow 0$, the particles become spatially decorrelated and the formulation by Griffa (1996) is recovered:

$$dx = \sigma \sqrt{dt/2} dw_0, \quad \left( \frac{dx}{dt} \right)^2 = \sigma^2.$$

(11)

The uncorrelated noise implies an independent forcing for each particle, leading to an infinite Lyapunov exponent (Piterbarg, 2001). This is due to the higher sensitivity of particle pairs to their differential random kicks as $\delta \ll L_K$ and $\delta \rightarrow 0$. The role of the parameter $D$ is to prevent the cumulative effect of uncorrelated random kicks to pairs of particles in close proximity, namely to constrain the second particle of each pair to take into account the random increment of the first particle.

### 3.2. LSGS-2 model

The LSGS-2 is a first-order Markov model that is joint Markovian in both the distance and the velocity (random flight), and introduces a decorrelation time scale $\tau$ for the turbulent velocity. This Lagrangian stochastic model is extensively used in the oceanographic community for realistic single-particle dispersion problems (e.g., Falco et al., 2000; Paris et al., 2005; Cowen et al., 2006; Paris et al., 2007). As such, it is important to include this model in our investigation of how different stochastic models affect relative dispersion.

The turbulent velocity increment along a particle trajectory in each direction is given by

$$du' = -\frac{\bar{u}}{\tau} dt + \sigma \sqrt{2dt/\tau} dw_0,$$

(12)

where $\tau \gg dt$.

Since the diffusivity is $K \sim \sigma^2 \tau$ for both LSGS models, and the decorrelation time scale for the random walk is half the integration time step, the velocity fluctuations of the random flight have to be much smaller than those of the random walk for the same diffusivity:

$$\sigma_{RF} \sim \sigma_{RW} \sqrt{\frac{dt}{2\tau}}.$$

(13)

Therefore a turbulent velocity fluctuation of 0.02 m/s in LSGS-2 has the same diffusivity as a 0.1 m/s one in LSGS-1 for a reasonable decorrelation time scale of $\tau = 1 \text{ day}$.

---

Fig. 5. The variations in scale-dependent FSLE $\lambda(\delta)$ (left panel) and relative dispersion $D^2(t)$ curves from HYCOM 1/12º with LSGS-1 random walk for different values of $L_K$. The dashed gray line is the approximate curve obtained by Lumpkin and Eliot (2010) from in situ drifter pairs, while the results from 1/48º HYCOM are shown in brown. Ballistic ($\lambda \sim \delta^{-1}$ and $D^2 \sim t$), Richardson ($\lambda \sim \delta^{-2/3}$ and $D^2 \sim t^{2/3}$), and diffusive ($\lambda \sim \delta^{-1}$ and $D^2 \sim t$), regimes are indicated in the background.
The spatial correlation is implemented in a similar way as for LSGS-1 by imposing the constraint $D^2$ on the turbulent velocity of the satellite particle.

3.3. LSGS-3 model

This model has been introduced by Haza et al. (2007b) in order to correct single-particle statistics and was applied in the context of a realistic NCOM simulation of the Adriatic Sea. It has not been applied to problems involving spreading of tracers, until now. LSGS-3 has a different conceptual approach with respect to LSGS-1 and LSGS-2. In the latter models, the turbulent velocity components that are added as stochastic perturbations to the trajectory equation are aimed to capture the entirety of turbulent processes independently of the local OGCM velocity field. On the other hand, the correction in LSGS-3 is based on the discrepancy between the statistical properties of OGCM-based and real/target trajectories, and the correction is highly dependent on the local velocity field.

Given the Lagrangian transport equation:

$$\frac{dx_m}{dt} = U_m(t, x_m) = U_m(t, f(x_m)) + u_m'(t, f(x_m)).$$  \hspace{1cm} (14)

where subscript $m$ denotes model, $U_m$ is the filtered model velocity representative of long-living mesoscale fields, and $u_m'$ is the remaining turbulent component in the submesoscale range, the purpose is to find corrected (denoted with subscript c) trajectories from:

$$\frac{dx_m}{dt} = U_m(t, x_c) + u_m'(t, x_c) + \eta(t).$$ \hspace{1cm} (15)

where the total corrected turbulent velocity component is $u'_c = u'_m + \eta$. The missing/corrected component $\eta$ is built in such a way to correct the main parameters of the model flow field, $\sigma_m$ and $\tau_m$, making them closer to the realistic/true ones, $\sigma_r$ and $\tau_r$, that are assumed known. The transport equation for $\eta$ can be written in two separate identical equations in the absence of explicit vorticity in the LSGS. The LSGS-3 formulation can be generalized to include horizontal vortices (Haza et al., 2007b) through the spin parameterization (Veneziani et al., 2005a). Here, we assume that some of the rotating component is already included in the resolved mesoscale eddies. This has the advantage of simplifying the LSGS, de-coupling the two velocity components, while avoiding the introduction of an additional (spin) parameter to be tested and tuned. The equation for the zonal component of $\eta$ then becomes:

$$\frac{d\eta(t)}{dt} = a \frac{du'_m(t, r_c(t))}{dt} + b u'_m(t, r_c(t)) + c \eta(t),$$ \hspace{1cm} (16)

where

$$a = \frac{\sigma_r/\sqrt{\tau_r}}{\sigma_m/\sqrt{\tau_m}} - 1, \quad b = \frac{\sigma_r}{\sigma_m/\sqrt{\tau_m}} - \frac{1}{\tau_r}, \quad c = \frac{1}{\tau_r},$$ \hspace{1cm} (17)

with $(\sigma_r, \tau_r)$ and $(\sigma_m, \tau_m)$ being the realistic/true and model velocity fluctuations and Lagrangian correlation times, respectively. These are the only parameters of LSGS-3. The variance of the missing component $\eta$ is given by:

$$\eta^2 \equiv (\sigma_r/\sqrt{\tau_r} - \sigma_m/\sqrt{\tau_m})^2 + (\sigma_r/\sqrt{\tau_m} - \sigma_m/\sqrt{\tau_m})^2.$$

which is zero if the model and the real parameters coincide.

Two important points need to be made here. The first is that while the mathematical derivation of LSGS-3 is based on a first-order Markov model, the final form contains no random forcing, with the exception of the initial random missing component $\eta_0 = \eta(t_0)$ for each particle, where $t_0$ is the time of launch.

Second is that the computation of the turbulent model component $u'_m$ requires a careful filtering of the model velocity fields on the order of the mesoscale eddy turn over time scale for the decomposition in Eq. (14), and it needs to be made sure that the performance of the LSGS-3 is robust to this procedure.

Finally, the correlation coefficient between the modeled and corrected velocities is:

$$r_{mc} = \frac{2\sqrt{\gamma}}{1 + \gamma},$$ \hspace{1cm} (19)

where $\gamma = \tau_m/\tau_r$, leading to values above 80% for $1/4 \leq \gamma < 4$, and to maximal correlation if $\tau_r = \tau_m$. Therefore, a vital albeit indirect consequence of the LSGS-3 model is that the spatial continuity of the detrended velocity field tends to be maintained by the high degree of correlation between the model and corrected turbulent velocities. This can be contrasted with the effects of the LSGS-1 and LSGS-2 models, that introduce an independent stochastic turbulent velocity simply superimposed to the model velocity.

4. Results

Our main objective is to better approximate $\lambda(\delta)$ curves from 1/48° HYCOM and observations from Lumpkin and Elipot (2010) by including LSGS models in particle advection using fields from 1/12° HYCOM.

4.1. Results from LSGS-1 model

LSGS-1 without space correlation: The effect of the LSGS-1 without space correlation, the pure random walk model, on the FSLE is computed and shown in Fig. 5a, where a choice of four $L_K$ in the range of 140 m $\leq L_K \leq 2.1$ km are used in conjunction with the trajectories from 1/12° HYCOM output. While $\lambda$ is unchanged for $\delta > 100$ km, it is dramatically enhanced in the submesoscale range. The range of scales influenced by the random walk increases with the amplitude of the kick. For example, a random kick with a standard deviation of 140 m during an advection time of 2 h impacts the relative dispersion at the scales up to 500 m only, while a kick with a standard deviation of 2.1 km will alter the relative dispersions at scales up to 100 km, thereby altering the entire exponential plateau, and the FSLE at the initial separation distance increases by a factor of 20. All curves with the exception of $L_K = 140$ m display an apparent power-law at the submesoscales, and gradually converge to the model FSLE at the mesoscale range. The power law at the small scales does not seem to depend on the amplitude of the random walk, with $\lambda \sim \delta^{-0.7}$ this result is surprising, considering the diffusive regime of Brownian motions, and we would have expected a $-2$ slope instead.

$$D^2(\delta) \text{ (Fig. 5(b))}$$ confirms most of these tendencies, since the Richardson power law of $t^2$ is maintained for $t \geq 5$ days. $D^2(\delta)$ with $L_K = 2.1$ km is the exception, since the range of spatial scales influenced extended to the mesoscales. For significant $L_K$ amplitudes, the small time exponential regime is replaced by a clear submesoscale regime mostly dominated by the random walk.

To isolate the effect of the mesoscale flow on the submesoscale relative dispersion, the particle ensemble is advected without the HYCOM velocities and $\lambda(\delta)$ is recomputed. As shown in Fig. 6a, the model’s contribution for small $\delta$ is negligible up to about $\delta = 5L_K$. Then the curves diverge to $\lambda \sim \delta^{-1.5}$ independently from $L_K$. While the relative dispersion of the random walk follows a $D^2 \sim t$ power-law (not shown), it did not translate into the expected $\delta^{-2}$ power-law even at the large scales, although it appears to asymptote to the $-2$ slope, if the particles had been advected over a much longer time. As such, the natural bias of the FSLE metric towards the fastest dispersive pairs end up describing regimes at all spatial scales more dispersive than what the dimensional arguments connecting $D^2$ to $\lambda$ would have predicted.
The trend of the FSLE at very small scales can be explained by separating the model and random walk components as follows. At the small scale $\delta$, $\lambda = \log(\alpha)/\tau$. Let’s define $V_{\text{total}}$ as the ensemble relative velocity of particle-pairs separating from $\delta$ to $\alpha \delta$. Then:

$$V_{\text{total}} = \frac{\langle x - x' \rangle}{\tau} = V_M + V_{RW},$$

where $V_M$ and $V_{RW}$ are the model and random-walk ensemble relative velocities, respectively. At the small scales, $V_M \ll V_{RW}$ and $\tau \sim O(\Delta t)$. We can then ignore the model contribution, and assume that only a limited number of time increments are required for the particles to separate from the distances $\delta$ to $\alpha \delta$. We then define $\tilde{u} = dx/dt$ as the random velocity generated at each time step, and $\sigma_a = L_K/\Delta t$ is the standard deviation of the random velocity fluctuations. Then the ensemble relative velocity from the pure random walk should be:

$$V_{RW} \sim \langle \tilde{u}^2 \rangle^{1/2} \sim \langle |\tilde{u}|^2 \rangle^{1/2}.$$  

Therefore:

$$V_{RW} \sim \sigma_a,$$

and

$$V_{\text{total}} = \frac{\langle x - x' \rangle}{\tau} \sim \sigma_a.$$  

This expression leads to:

$$\frac{\log(\alpha)}{\tau_a} \sim \langle \tilde{u}^2 \rangle^{1/2} \sim \langle |\tilde{u}|^2 \rangle^{1/2}.$$  

or

$$\lambda \sim \frac{\log(\alpha)}{\tau_a} \sim \frac{\sigma_a}{\delta}.$$  

This expression is a function varying like $\lambda \sim \delta^{-

1}$. The coefficient of the power law is $L_K$, the amplitude of the Brownian motion, which explains the shift of $\lambda$ at the small scales to the right, when we increase $L_K$. Given that an excellent agreement is obtained between the predictions from (25) and the computed FSLE curves in Fig. 6b, the conclusion that the LSGS-1 affects the FSLE at scales much larger than $L_K$ is confirmed.

It is widely believed that in the case of a random walk model, the FSLE is proportional to $Kt^{-2}$, where $K$ is the diffusivity. In fact, this is just a limit as $\Delta t$ goes to zero in the following set up. Independent random displacements (kicks) of two particles occur in discrete moments $n\Delta t$, $n=1,2,3,...$, and have variance of order $K\Delta t$. If $\Delta t$ is finite, then the regime $\delta^{-1}$ is possible. We illustrate this by a simple but realistic example of a mathematical model in the Appendix A.

As shown in Fig. 6(b), the shape of $\lambda$ for a pure random walk is very similar to the analytical results of a discrete time and continuous jump distribution with a double-exponential pdf (Piterbarg, 2011). Note also that $\lambda$ does not diverge as $\delta$ tends to zero, but seems to asymptote instead to 1 in this non-dimensional form, thus confirming the analytical formulation (A.4) in the Appendix A. Finally, and as highlighted by the linear fit, the effect of this type of random walk translates into a constant relative velocity at the small scales.

On the other hand, $D^2(t)$ of the pure random walk is strictly linear, and is defined as:

$$D^2(t) = D_0^2 + 2Kt = D_0^2 + \frac{L_K^2}{\Delta t}.$$
Note that since $D^2_2$ is non-negligible, the linear trend is not necessarily apparent on a log–log scale, which can mask the signature of a diffusive regime if it happens to dominate the early stage of the relative dispersion, as it is the case here. For weaker random noises (such as $L_k \leq 500$ m), the curve might even be confused with an exponential regime.

The signature of a stochastic process has been distinguished previously from deterministic chaos by Gaspard and Wang (1993) using the $\langle x , x \rangle$-entropy metric, Cencini et al. (2000), Gao et al. (2006) using the scale-dependent Lyapunov exponent (SDLE). Both metrics are similar to the FSLE in that they measure the amount of information of a signal at a given scale, and thus are a perfect tool for isolating different types of signals and/or regimes. Gao et al. (2006) also remark the shift of the curve to the right when the amplitude of the random walk is increased, up to the critical case where differentiation is no longer possible. In the example we showed in Fig. 5, it corresponds to $L_k > 2$ km.

Note that a similar trend was obtained by Lumpkin and Elipot (2010) with in situ surface drifter pairs released in the Gulf Stream. The FSLE revealed a clear power-law regime for the scales $2$ km $\leq \delta \leq 10$ km, and values close to the ones we obtained with HYCOM 1/12° and $L_k$ of about 1–2 km. It would correspond to random velocity fluctuations of about 15–20 cm/s, which are not unreasonable. A $\delta^{-1}$ regime was also obtained from a numerical simulation of the tidal flow through Moskstraumen (Norway) by Lynge et al. (2010), with a very high resolution of 50 m, extending to 5 km.

**LSGS-1 with space correlation:** Constraining the random noise to be spatially correlated leads to a different dispersion regime in the small scales $\delta \leq 2 R_0$. Fig. 7a shows the FSLE of the 1/12° trajectories with random walks of $L_k = 1$ km and $L_k = 2.1$ km, modulated by different spatial correlations in the scale range of 500 m $\leq L_0 \leq 20$ km. A 2.5 km decorrelation scale already reduces the smallest scale FSLE from $4^{-1}$ to $1.5^{-1}$ for $L_k = 1$ km, and the effect is felt up to $\delta = 5$ km, with a maximum attained at $\delta \approx 4$ km. For $L_k = 10$ km, the smallest scale FSLE is now equal to the model’s exponential value, and $\lambda$ reaches a maximum at $\delta \approx 15$ km. The type of spatial correlation used in this case affects the FSLE at the smallest scales, and reaches a local maximum at $\delta \approx 1.6 L_k$, before converging on the pure random-walk curve. Note that the effect of the random walk is still felt at the intermediate scales of $L_k < \delta < 2 R_0$, in particular if $L_0 \ll R_c$.

A good match with the observed relative dispersion from Lumpkin and Elipot (2010) is obtained with the parameter combination of $L_k = 2.1$ km and $L_0 = 500$ m. The value of $L_0$ needed to attain this match is perhaps indicative of the scale of submesoscale eddies.

Using the same arguments as in the pure random-walk case, the separation of model and noise contributions yields in this case:

$$V_{RW} \sim \left< |v_2 - v_1|^2 \right>^{1/12}, \quad \text{with} \quad v_2 = Dv_1 + \left(1 - \frac{1}{\delta^2}\right) \frac{dw}{dt},$$

so that:

$$V_{RW} \sim (1 - D) \left< |v'|^2 \right>^{1/12} \sim (1 - D) \sigma^2,$$

and

$$\lambda \sim \left\{ \frac{\log(\sigma)}{(x-1)dt} \right\}^{1/2} \frac{L_k}{\delta} \left(1 - e^{-\frac{L_k}{\delta}}\right).$$

Therefore, at the scales $L_0 \leq \delta \leq 2 R_0$, the FSLE curve is shaped by the function $\left(1 - e^{-\frac{L_k}{\delta}}\right)/\delta$, with a local maximum corresponding...
to its zero derivative with respect to \( \delta \) (Fig. 7b). It is the non-trivial solution of the equation:

\[
e^{\delta^2} - 2u - 1 = 0, \quad \text{with} \quad u = \frac{\delta^2}{2\tau},
\]

leading to:

\[
u_0 = 1.26, \quad \text{and} \quad \delta_0 \approx 1.59 L_p.
\]

Therefore, the local maximum depends only on the decorrelation length scale.

The influence of LSGS-1 on the transport barriers and stretching rate of the surface flow is shown in Fig. 8. With a parameter of \( L_p = 1 \) km (and \( L_0 = 0 \)) the FSLE ridges become grainy, although their coherence is maintained, and in fact their amplitude is increased. The effect is significantly softened with a slight spatial correlation constraint of \( L_p = 2.5 \) km. A recent work by Carrasco et al. (2011) also points to the robustness of the mesoscale FSLE ridges to the inclusion of stochastic noise.

4.2. Results from LSGS-2 model

Computations of \( \lambda(\delta) \) for different statistical parameters (Fig. 9a) show that the LSGS-2 model has a very similar impact to the small scale dispersion as the LSGS-1 model. The parameter \( \sigma \) appears to have a dominant influence, and we note that their values for the same rate of dispersion as the random walk and a typical decorrelation time scale of \( \tau = 1 \) day are about 5–10 times smaller, as expected for a constant diffusion coefficient.

There is a slight difference in the power-law in the submesoscale range, with \( \lambda \sim \delta^{-0.78} \) and this is probably the contribution of the memory term. This results in a flatter curve, or a smoother transition from the submesoscale to the mesoscale regime. Increasing the decorrelation time scale appears to reduce \( \lambda_{\max} \) marginally. The flattening tendency of \( \lambda \) near the initial particle pair distance of 1 km is actually imposed by \( dw_\delta(0) = 0 \), leading to a smaller rate of dispersion than the full uncorrelated random walk. As expected, the introduction of spatial correlation further reduces the FSLE at the small scales and shifts the local maximum to a larger \( \delta \) according to a constraining function similar to the LSGS-1.

\( D^2(t) \) also benefits from the slow LSGS contribution initially (Fig. 9b), and the departure from the model appears after a few iterations. However, the Richardson regime is present after 1 day, with the exception of the high \( \sigma = 14 \) cm/s experiment, also corresponding to the absence of the mesoscale kink in the FSLE. The transition to the diffusive regime occurs earlier for large values of \( \sigma \), since the LSGS model is more dominant.

The experiment with \( \sigma = 7 \) cm/s leads to the closest approximation to the \( D^2(t) \) curve from HYCOM/148°

4.3. Results from LSGS-3 model

Flow decomposition: The LSGS-3 should satisfy the following two conditions. First, \( \vec{u}_c \ll \vec{U}_{so} \), and second, \( \lambda(\delta) \) should remain the same for \( \delta > 2 \vec{R}_g \). This condition is paramount since it implies that the particle motions are still constrained by the flow transport barriers defined by the eddies of scales of \( \vec{R}_g \). Both conditions imply that all Gulf Stream features such as meanders and eddies which all have typical velocity scales of 1 m/s should be included in the deterministic drift \( \vec{U}_{so} \).

In our previous work (Haza et al., 2008), we considered two kinds of low-pass filters for the surface velocity field: spatial and temporal averages. It was found in a high-resolution simulation of the Adriatic Sea, that a simple spatial Gaussian weighted average with scales as small as 5 km reduced relative dispersion at scales up to 100 km (5–10 times local \( \vec{R}_g \)). However, temporal filtering based on a simple time-moving average had only a limited impact; a time window of 30 days affects the relative dispersion at scales of up to 20 km at the most. This is due to the stronger impact spatial smoothing has on the coherent structures of the mean flow field, by altering their horizontal strain, thus changing the relative dispersion at much larger separation scales. It is thus preferable to select a temporal filter, with a time-window large enough to leave room for \( v \) modifications, while preserving the stretching rate of the mesoscale features.

In the case of the Gulf Stream, the mesoscale coherent structures not only have larger length scales, but they are also more energetic than the mesoscale features of the Adriatic. A low-pass window of 20 days was found to preserve their coherence, thereby not influencing the relative dispersion at scales larger than 100 km, corresponding to the critical length scales of 2–3 \( \vec{R}_g \).

The auto-covariance functions for \( u \) and \( v \) computed from the trajectories after detrending with the 20 day low-pass filter (hereafter LP20) are displayed in Fig. 10. The resulting statistical parameters are listed in Table 1. The correlation time scales in both zonal and meridional directions are about 1.5 days, which is in reasonable agreement with previous parameter estimates in this region (Garraffo et al., 2001b; Veneziani et al., 2005a). The standard deviations of the turbulent velocity fluctuations are less than 20 cm/s, or about one order of magnitude smaller than the typical Gulf Stream eddy velocities. In the case of the 5 day low-pass filter (LPS), both velocity fluctuations and decorrelation timescales are
Fig. 9. (Left panel) Variations in scale-dependent FSLE $\lambda(\delta)$ curves from HYCOM 1/12° with LSGS-2 model for different values of $\sigma$, $\tau$, and $L_D$. (Right panel) Sensitivity of $D^2(t)$ to $\sigma$ is displayed for $\tau = 1$ day, $d_0 = 2.4$ km and $dt_{0}(0) = 0$.

Fig. 10. Autocorrelation functions of trajectories from HYCOM 1/12° and 1/48° output after LP20 and LP5 detrending.

reduced by a factor of $\sim 1.5$, with $\sigma_m \sim 11$ cm/s and $\tau_m \sim 1$ day, which is expected since more momentum is assigned to the mean flow due to the shorter temporal averaging. There is no significant difference between these parameter estimates obtained from 1/12° and 1/48° flow fields.

The performance of the LSGS-3 model from the standpoint of one-particle statistics can be evaluated in two ways. First, by comparing the absolute dispersion $\rho(t)$ of the corrected trajectories to that of the model trajectories at large times, which is expected to change according to the ratio $\sigma_m \sqrt{\tau_c}/\sigma_m \sqrt{\tau_m}$. Second, by computing the auto-covariance functions of $u$ and $v$ from the corrected trajectories, and see how close the corrected parameter pair ($\sigma_c, \tau_c$) is to the target/real pair ($\sigma_r, \tau_r$).

An ensemble set of arbitrarily given ($\sigma_c, \tau_c$) parameters were used to test the LSGS-3 model with the surface Gulf Stream in HYCOM 1/12°. The absolute dispersion is computed and displayed in Fig. 11a. For each set of parameters, the ensemble of particles were launched at three different times to check the stability and validity of the results.

As indicated by the rainbow colormap, we would expect $\rho(t)$ to reach significantly higher values than the model for a high ($\sigma_r \sqrt{\tau_c}/\sigma_m \sqrt{\tau_m}$)-ratio, which is what we observe (orange, red curves), with the exception of a single realization. $\rho(t)$ is also reduced successfully to values below 1/12° HYCOM for ($\sigma_r \sqrt{\tau_c}/\sigma_m \sqrt{\tau_m}$) < 1, corresponding to the cases where the LSGS-3 decreases the Lagrangian turbulent velocity component (dark blue curves).
The proximity of the corrected parameter set \((r_c, s_c)\) to the target parameter set \((r_r, s_r)\) is illustrated in Fig. 11(b) and (c) in both zonal and meridional directions. The LSGS model appears to perform best (i.e. by how close \((r_c, s_c)\) is to \((r_r, s_r)\) on the scatter-plot) when only \(r\) is modified. This is consistent with our previous finding in the Adriatic Sea application (Haza et al., 2007b). The error increases with the difference between the real parameters and the model parameters, in particular when this difference is combined by both \(r\) and \(s\) ratios, the highest here being for \((r_r/s_r = 2, \tau / \tau_m = 4)\). The mean flow nevertheless is also adding to the discrepancies.

Overall, it can be stated that the LSGS-3 tends to perform satisfactorily in most cases, especially when considering how far apart the Gulf Stream dynamics is from pure homogeneous turbulence with zero-mean flow.

An interesting finding is that we note from Table 1 that the Lagrangian parameters are the same for both resolutions of HYCOM, subject to the same temporal filtering. While the FSLEs from the two model outputs differ significantly at the submesoscales, it is clear that the submesoscale features resolved by 1/48° do not contribute to a significant change in the parameters of the autocorrelation function \((\sigma_m, \tau_m)\), which is a single-particle metric. This result implies that the fluctuation velocities are therefore dominated by the high kinetic energy of the mesoscale residual motions.

Indeed, the relatively weak and short-lived submesoscale motions are imbedded in the strong and long-lived mesoscale motions and are transported by the mesoscale eddies, exerting only minor influence on the single particle trajectories. Nevertheless, submesoscale eddies and filaments clearly contribute to a faster separation of particles from one another.

**Effect of the LSGS-3 on the relative dispersion:** The influence of LSGS-3 for different parameters in the range of \(1/2 \leq r_r/s_r \leq 3\) and \(1/4 < \tau / \tau_m < 4\) on \(\lambda(t)\) and \(D^2(t)\) is presented next. Either \(r_r/s_r\) or \(\tau / \tau_m\) is changed, not both simultaneously in this set of experiments. This is carried out for both LP20 and LP5 temporal filtering of HYCOM 1/12° field.

The results obtained by varying \(r_r/s_r\) are shown in Fig. 12. The parameter \(r_r/s_r\) appears to affect the relative dispersion by shifting the FSLE plateau towards a higher (lower) \(\lambda_{\text{max}}\) value when \(r_r\) is larger (smaller) than \(s_r\). With the exception of the \(r_r/s_r = 3\) case, the change in the FSLE for \(\Delta > 100\) km is not significant. These trends are reflected in the \(D^2\) curves, where the Richardson \(I^3\)
power law is attained for all cases except for $r/r_m = 3$. The shift in
time is due to the change in the exponential regime within the first
time. The effect of $s/r_s$ on the relative dispersion is less clear
at the small scales, in particular when $r/r_m = 1$. It does however
change $D^2$ at large times, by decreasing (increasing) it if $s$ is small-
ner (larger) than $s_m$.

The length of temporal filtering plays a role in the results. Note
that there is very little difference between the FSLE curves from
unfiltered HYCOM and with LP5 filtering (Fig. 12(b)). A similar va-

tue of $r/r_m$-ratio acts differently on trajectories filtered by LP5.
Not only is $k_{\text{max}}$ different when comparing with the LP20 experi-
iments, but also the range of spatial scales where the FSLE is af-
fected is reduced. This is in part due to the smaller kinetic
energy residual of the mesoscale structures present in the detrend-
ed field, being significant only at the scales below $R_d$. As a result,
the extent of the modified exponential regime is reduced to shorter
scales, and a smoother transition occurs at the intermediate scales.

A possible relation between the change in the FSLE plateau and
the parameter $s/r_s$ is explored next. Following the simplified
assumptions of the random walk case and the apparent fit, let’s as-
sume that in the exponential regime, the model relative velocity
covering the distance $(x - 1)\delta$ is $\Delta V_m \sim \Delta V_{LP} + \Delta V_{rv}$, and the cor-
rected relative fluctuation velocity $\Delta V_r \sim \left(\frac{s}{s_m}\right)^{-1} \Delta V_m$. Then

Fig. 12. Sensitivity of $\lambda(\delta)$ (upper panels) and $D^2(t)$ (lower panels) to different values of expected $s_r$ and $t_r$. For LP20 (left panels) and LP5 (right panels) temporal filtering.

Fig. 13. Log-log plot of $(k_c/C_0 k_{LP})/(k_m/C_0 k_{LP})$ as a function of $r/r_m$ from Eq. (34). The
slope is $\gamma = 1.9$. 

A.C. Haza et al. / Ocean Modelling 42 (2012) 31–49
\[ \frac{1}{\tau_m} - \frac{1}{\tau_{LP}} = \frac{1}{(x - 1)\delta} (\Delta V_m - \Delta V_{LP}) = \frac{\Delta V'_{LP}}{(x - 1)\delta}. \tag{32} \]

and

\[ \frac{1}{\tau_c} - \frac{1}{\tau_{LP}} = \frac{1}{(x - 1)\delta} \left( \frac{\sigma_c}{\sigma_m} \right)^{\gamma} \Delta V_m = \left( \frac{\sigma_c}{\sigma_m} \right)^{\gamma} \left( \frac{1}{\tau_m} - \frac{1}{\tau_{LP}} \right). \tag{33} \]

Therefore:

\[ \lambda_c \sim \lambda_{LP} + \left( \frac{\sigma_c}{\sigma_m} \right)^{\gamma} (\lambda_m - \lambda_{LP}). \tag{34} \]

where the right hand side of Eq. (34) is known, except that \( \gamma \) is a regression parameter. \((\lambda_c - \lambda_{LP})(\lambda_m - \lambda_{LP})\) is shown in a log–log plot (Fig. 13). This plot yields \( \gamma \approx 1.9 \), as well as a collapse of curves from LP20 and LP5 filtering.

The simple direct relation between \( \Delta V'_{LP} \) and \( \Delta V'_m \) makes sense when we consider the correlation coefficient between the modeled and corrected fluctuation velocities, which is expected to be maximal when \( r \) is fixed. In the statistical context, both components of the corrected velocity fluctuations tend to vary like \( u'_c(t) = \sigma_c u'_m(t) \) along the particle trajectory.

The influence of the LSGS-3 on the FSLE maps, as well as Okubo-Weiss maps are shown in Fig. 14 as a function of varying \( r/r_m \) in LSGS-3.

Fig. A.14. Changes in the FSLE (left panels, color scale is in 1/day) and Okubo–Weiss (right panels, color scale is in 1/day²) maps from the reconstructed velocity fields of the trajectories near their initial launch time as a function of varying \( \sigma_c/\sigma_m \) in LSGS-3.
and is consistent with
This case indicates that the separation of scales no longer holds,

Summary of the LSGS model specifics and their impact on the submesoscale (SMS) dispersion.

The experiment
the grid size, unlike the case with the LSGS-1 and LSGS-2 models.

the grid size, unlike the case with the LSGS-1 and LSGS-2 models. The experiment \( \sigma_j/\sigma_m \geq 2 \) (bottom left panel) seems to reach a critical point beyond which the mesoscale ridges become affected (for fixed \( \tau \)) by substantially modifying their spatial distribution. This case indicates that the separation of scales no longer holds, and is consistent with \( \lambda(2\sigma_m,2R_d < \delta < 3R_d) \) being significantly higher than the HYCOM 1/12° counterpart in Fig. 12 (upper-left panel red curve).

As highlighted in the Okubo-Weiss maps, increasing the magnitude of the velocity fluctuations leads to an increase of the stretching rate in the regions where the shear was already significant in the model, such as the edges of the Gulf Stream meanders. Other less energetic regions, where the shear and vorticity are absent, remain unchanged. This is unlike the other two models, in which velocity perturbations independently of the model fields are added to the Lagrangian position increments.

Next, the sensitivity of \( \lambda \) and \( D^2 \) to variations in \( \tau_\ell/\tau_m \) is investigated for a fixed value of \( \sigma_j/\sigma_m \). This latter parameter is taken as \( \sigma_j/\sigma_m = 2 \), since that leads to a quite close agreement with the FSLE curve from the 1/48° resolution output (Fig. 15). Nevertheless, while the magnitude of \( \lambda_{\text{max}} \) is quite close, these curves differ in the intermediate range of 30 km \( \leq \delta \leq 300 \) km. It is found that the second control parameter helps improve the agreement between LP20-filtered 1/12° output with LSGS-3 and 1/48° result for this particular scale range, for instance by using \( \tau_\ell/\tau_m = 1.5 \) and \( \tau_r/\tau_m = 2.0 \), but this comes at the expense of reduced \( \lambda_{\text{max}} \).

The effect is also quite clear for \( D^2(t \geq 30 \) days) where \( \tau_r/\tau_m > 1 \) leads to an increase in the relative dispersion, and consistent with the increase in the absolute dispersion. Conversely, \( \tau_r/\tau_m < 1 \) shifts the exponential regime to higher hyperbolicity. A possible interpretation for this trend at the small scales might be that particles decorrelating at a faster rate can emulate the impact small-scale features would have on the relative dispersion.

5. Discussion of performance characteristics of LSGS models

The differences between the three LSGS model results are highlighted in Table 2 and examples of parametrized trajectories are displayed in Fig. 16. The coherence of the mesoscale transport barriers is evaluated both qualitatively from their degree of visibility, and quantitatively from the transition of the scale-dependent FSLE-curve at the scales \( R_d < \delta < 3R_d \).

In the case of both LSGS-1 and LSGS-2, the mean flow is the ocean model velocity field, while the turbulent component is entirely parametrized. Parametrizing the submesoscale transport by an uncorrelated Gaussian noise (LSGS-1) is shown here to leave a distinct signature in the FSLE metric at the scales below the radius of deformation: for a reasonable standard deviation of only 10% of the typical velocity scales in the region, the rate of dispersion is increased by a factor of 10 at the model grid-scale, and the exponential regime is replaced by a power-law of \( \delta^{1.2} \) up to the \( R_d \).

Using LSGS-1 and LSGS-2, the computation of the scale-dependent FSLE not only reveals values much higher than the model’s exponential regime at the submesoscales, but also the existence of power-law regimes that do not match the expected \( 2 \) slope for the diffusive regime. The departure from theory appears to be related to the type of random walk obtained from continuous jump distribution with finite time increments \( \Delta t \), leading to a transition at small \( \delta \) from a \( \sim \delta \) slope to the classic \( \sim \delta^{-2} \) slope. The \( \delta^{-1} \) regime corresponds also to a constant average relative velocity at the small scales, i.e. independent of \( \delta \) and proportional to the standard deviation of the random velocity fluctuations \( L_k/\Delta t \). Since this power law is also indicative of the ballistic regime, other complementary metrics (such as \( D^2(t) \)) must be used to differentiate between the two.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Summary of the LSGS model specifics and their impact on the submesoscale (SMS) dispersion.</th>
</tr>
</thead>
<tbody>
<tr>
<td>LSGS-1</td>
<td>LSGS-2</td>
</tr>
<tr>
<td>Mean flow V</td>
<td>WCOM/12°</td>
</tr>
<tr>
<td>( \rho(V, \dot{V}) )</td>
<td>0 (Uncorrelated)</td>
</tr>
<tr>
<td>SMS FSLE regime</td>
<td>( \lambda \sim \delta^{1.1} )</td>
</tr>
<tr>
<td>Spatial impact</td>
<td>Uniformly</td>
</tr>
</tbody>
</table>

A.C. Haza et al. / Ocean Modelling 42 (2012) 31–49 45

Author's personal copy
On the other hand, the LSGS-3 relies on a modification of the model’s turbulent Lagrangian velocities by changing their statistical parameters \((\sigma_m, \tau_m)\) to targeted ones \((\sigma_r, \tau_r)\) under the Markov-1 assumption. This implies that the full model velocities cannot be used as the mean flow. Instead, the deterministic drift is obtained from a local temporal low-pass filter so as to preserve most of the horizontal shear from the mesoscale eddy field. Implementation of the LSGS-3 model on the Gulf Stream circulation with a 5–20 day low-pass has resulted in a non-local regime below the \(R_0\) which has a higher (lower) hyperbolicity when the ratio of the turbulent velocity fluctuations \(\sigma_r/\sigma_m\) is increased (decreased), while the rate of dispersion at the larger scales is unchanged. Modifying \(\tau\) on the other hand can affect a wider range of scales, including the regime transition at the scales around 2–3\(R_0\). Generally, changing \(\tau\) has an opposite impact on the small-scale relative dispersion with respect to the large time dispersion. For instance, a decrease in \(\tau\) (i.e. particles decorrelating at a faster rate) can have an similar impact as small-scale features would have on the relative dispersion, and leads to an increase in \(\lambda_{\text{max}}\). Increasing both model parameters allowed for a very close reproduction of the 1/48° HYCOM \(\lambda\)-curve.

The impact of the different LSGS models on the trajectory ensemble is illustrated in Fig. 16, where one can see that the changes in particle evolution and distribution from LSGS-2 is barely visible, while LSGS-3 acts to expand their spatial coverage to the extent that more than half of the particles ended up evolving along the northward branch defined by the mesoscale transport pathways. Visual comparison with the spatial distribution of the 1/48° HYCOM trajectories indicates a similar pattern of dispersion, and it can be quantified by the second moment of particle displacement \(\sigma_{CM}(t)\). As shown in Fig. 17, LSGS-1,2,3 enhance \(\sigma_{CM}(t)\) with increasing intensity (a result already observed by Ullman et al. (2006) for the random walk and random flight). A direct comparison of the three LSGS models with 1/48° HYCOM clearly indicates that the substantially higher \(\sigma_{CM}(t)\) value due to the impact of the submesoscale features on the relative dispersion at the small scales can be attained by the LSGS-3 model when doubling the model parameters.

6. Summary and conclusions

Particle dispersion computed from eddy permitting ocean models is generally characterized by a lack of dispersion at submesoscales. However, recent observations and multi-resolution modeling studies reveal increasing dispersion rates at the submesoscale range. This indicates that the models have a tendency to underestimate the submesoscale strain rate. While submesoscale motions have a significant impact on the Lagrangian transport, they are computationally challenging to resolve explicitly.

Dynamical systems methods to compute LCS have been proven very useful in order to characterize the influence of long-lasting, slowly-varying mesoscale features on ocean transport, but these methods require much more extensive data sets and high resolution accurate computations to provide further insight into complex rapidly-evolving submesoscale flows. On the other hand, it has not been straightforward to incorporate mesoscale features in LSGS models, but these statistical methods appear more suitable as flows become increasingly more turbulent.
We put forward the hybrid approach of combining LSGS in the submesoscale range and LCS from realistic OGCMs for the mesoscale range. The idea of using LCS and LSGS was also employed by Carlson et al. (2010) in the context of a smaller coastal application in the Gulf of Elat. Here, our main focus is in extending these ideas to open ocean flows. At the very core of this concept is the challenge of how to separate mesoscale and submesoscale motions. In particular, our investigation has been driven by the following questions:

- Can one combine LCS and LSGS methods in mesoscale eddy-resolving OGCMs in places which exhibit, in space and/or in time, strong submesoscale features, in order to attain and analyze realistic relative dispersion in the multi-scale setting of the ocean?
- Is it possible to adapt an LSGS model which would not only improve the scale dependent relative dispersion statistics, but also maintain the Lagrangian transport barriers of the mesoscale eddy field set by the ocean model?

In response to these questions, we find that adapting the previously developed LSGS models to the parametrization of submesoscale dispersion by shifting the separation of scales to the radius of deformation, and evaluating their performance under a scale dependent relative dispersion metric has resulted in a substantial improvement of submesoscale Lagrangian transport. The innovative aspect of this study has relied on the combination of single and dual particle statistics, which has allowed for a better monitoring/control on the scale dependent relative dispersion, as well as maintaining the coherence of the mesoscale transport pathways. Specifically, the scale-dependent FSLE metric is particularly suited to monitor over which scales of motion the LSGS models are acting and whether mesoscale dispersion characteristics are influenced or not.

We found that the three considered LSGS models have significantly different properties. While all three of them appear to enhance the small-scale strain level in OGCMs, LSGS-1 and LSGS-2 attain this by adding turbulent fluctuations everywhere, while the LSGS-3 model is spatially selective on which regions to influence. In particular, using LSGS-3 we see a high correlation between the Lagrangian turbulent velocities from parametrized and modeled trajectories which results in a smooth enhancement (or reduction) of the pre-existing mesoscale transport barriers, leading therefore to intensification (or weakening) of the model’s hyperbolic regions. LSGS-3 is perhaps the model to implement in regions where the FSLE follows a smooth transition from the small scale exponential regime to the power-law regimes starting at $2R_e$, with...
an observed to model ratio of 2–3 at the most for the Lyapunov exponent. Furthermore, its capability to enhance turbulence velocity fluctuations selectively in hyperbolic regions makes it a parameterization of choice for transport of submesoscale features, considering their tendencies to emerge by frontal instabilities. On the other hand, both LSGS-1 and LSGS-2 (with or without spatial correlation) can lead to an increase in $\lambda_{\text{max}}$ by a factor of 10 near the initial grid-scale separation, making them appropriate in cases of strong external forcing, as opposed to internal flow instabilities.

Nevertheless, it remains still difficult at this time to provide indications on which one of the three parametrizations is more realistic given the sparsity of in situ data present currently for validation. It is entirely possible that the choice of the appropriate LSGS will depend on the specific application considered. For the Gulf Stream region considered here, the results of Lumpkin and Elipot (2010) indicate a $\lambda \sim \delta^{-1}$ slope in the submesoscale range, similar to the results of LSGS1 and LSGS2. This in turn suggests that drifter motion falls into the same category of random walk increments with scales of the order of 1–2 km. On the other hand, it should be kept in mind that drifter launches in Lumpkin and Elipot (2010) occurred during the winter season, with high winds and waves, and were clustered in relatively small areas. It is therefore unclear whether they can be considered typical of the area, and should be kept in mind that drifter motion falls into the same category of random walk increments for general pdf $p(x)$ with support on $[0,\infty)$. It is then relatively straightforward to prove the possibility of existence of $\lambda_{\text{max}}$ in a certain range.

$\lambda(\delta) = \ln xT(\delta, 2\delta)$. 

It can be shown by standard probabilistic means that $T(x, r)$ satisfies the following integral equation:

$$T(x, r) = \int_{-r}^{r} p(x - y)T(y, r)dy + \Delta t. \quad (A.1)$$

This equation has an exact solution for the double exponential distribution:

$$p(x) = \frac{1}{2\sigma} e^{-|x|/\sigma},$$

which is very close to the Gaussian distribution, where $\sigma$ is the standard deviation of the relative displacement at each particular moment. Namely:

$$T(\delta, 2\delta) = \Delta t \left( \frac{(x^2 - 1)\delta^2}{\sigma^2} + \frac{2\delta}{\sigma} + 1 \right). \quad (A.2)$$

To analyze the corresponding expression for FSLE represent the variance $\sigma^2 = K\Delta t$ in terms of diffusivity and assume that:

$$\delta \gg \sqrt{K\Delta t}/x.$$}

Under this assumption the third term in (A.2) is much less then the second one and we get:

$$T(\delta, 2\delta) \approx \frac{(x^2 - 1)\delta^2}{K} + \frac{2\delta \sqrt{K\Delta t}}{K}.$$}

From this expression it follows that for:

$$\Delta t \ll \frac{(x^2 - 1)\delta^2}{x^2};$$

the second term is negligible and we get:

$$\lambda(\delta) \approx \frac{\ln x}{x^2 - 1} K\delta^{-2}.$$}

Thus, for small $\Delta t$ we have a regime which commonly viewed as universal. However in the opposite case (essentially discrete time):

$$\Delta t \gg \frac{(x^2 - 1)\delta^2}{x^2};$$

the second term dominates and we get:

$$\lambda(\delta) \approx \frac{\ln x}{x^2} \sqrt{K}\delta^{-1}. \quad (A.4)$$

Hence the $\delta^{-1}$ regime can be observed in the case of comparable $\Delta t$ and $\delta^2/K$ and even when $\Delta t < \delta^2/K$ if $x$ close to 1.

The above asymptotics are hardly applicable for a quantitative analysis of the FSLE regimes because of the one-dimensionality assumption and specific distribution of kicks, but they certainly prove the possibility of $\delta^{-1}$ regime.

Moreover, similar arguments lead to basically the same conclusions for general pdf $p(x)$ with $p(0) > 0$. In this general case $\sigma$ should be replaced by $1/[2p(0)]$, So the probability mass around zero plays the crucial role in explanation of the $\delta^{-1}$ regime.

Notice also that the assumption of continuous distribution for displacements (jumps) is very essential. For example, if particle jumps are exactly equal to $\sigma$ or $-\sigma$ with equal probabilities 1/2 (a simple random walk), then it can be shown that $\lambda$ is again proportional to $K\sigma^{-2}$ for the whole range of $\delta$. Thus, a discrete time and continuous distribution of jumps are necessary conditions for the existence of $\delta^{-1}$ regime.

Finally, in 2D case an equation similar to (A.1) can be derived, but its analysis is much more complicated. In particular no exact solution was found for it. However we managed to find an asymptotical solution for large $\sigma$ which also exposes a regime $\delta^{-1}$ in a certain range.

References

